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On the reduction methods for ordinary differential equations

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Abstract

A standard method based on the use of differential invariants of a Lie group, $\mathcal{G}$, enables us to reduce any ordinary differential equation invariant under the action of $\mathcal{G}$. We show that this method is applicable to vector fields more general than those associated with Lie symmetries. We characterize all such vector fields and study their relationship with nonlocal symmetries and λ-symmetries (Govinder K S and Leach P G L 1995 *J. Phys. A: Math. Gen.* **28** 5349–59, Muriel C and Romero L 2001 *IMA J. Appl. Math.* **66** 111–25).

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1. Introduction and background

Invariance under transformations, an essential feature of the mathematical description of physical phenomena, often enables the solutions to the equations of mathematical physics to be obtained by symmetry arguments [5, 18]. For example, a symmetry of a first-order ordinary differential equation leads to integration by *quadrature*, while a symmetry of a higher-order ordinary differential equation leads to a *reduction* of its order. If a sufficient number of symmetries of the *right type* is available then also higher-order ordinary differential equations may be solved by reduction and quadrature. (These classical results may be found in the monographs [5, 18] or in the introductory survey [13].)

Let us consider the nth-order ordinary differential equation

$$\Delta\left(x, u, u_{(n)}\right) = 0 \tag{1}$$

in the unknown u and independent variable x. Here $u_{(n)}$ denotes the set of all derivatives of u with respect to x up to an order n.

It is well known that the theory of differential equations may be approached also from a geometrical point of view. This is achieved by introducing the coordinates

$$q_1 \equiv \mathrm{d}u/\mathrm{d}x, q_2 \equiv \mathrm{d}^2u/\mathrm{d}x^2, \ldots, q_n \equiv \mathrm{d}^n u/\mathrm{d}x^n$$

and regarding $\Delta(x, u, q_1, \ldots, q_n) = 0$ as a manifold $\mathcal{S}$ in a suitable space. Any solution, $u = f(x)$, of (1) represents a curve $\mathcal{C}$ which is in $\mathcal{S}$, but not every curve in $\mathcal{S}$ corresponds to a solution of (1). Indeed, solutions of (1) are only those curves $\mathcal{C} \subset \mathcal{S}$ that at all points satisfy the differential relations

$$\mathrm{d}u = q_1\,\mathrm{d}x \qquad \mathrm{d}u_{(i-1)} = q_i\,\mathrm{d}x \qquad i = 2, \ldots, n. \tag{2}$$

Accordingly, equation (1) may be suitably regarded as a manifold in $X \times U^{(n)}$, the nth-order *jet-space* of the underlying space $X \times U$. The coordinates of $X \times U^{(n)}$ represent the independent variable, the dependent variable and the derivatives of the dependent variable up to n.

We consider a one-parameter Lie group, $\mathcal{G}$, of point transformations

$$\begin{aligned} x^* &= x^*(x, u; \epsilon) = x + \epsilon\xi(x, u) + O(\epsilon^2) \\ u^* &= y^*(x, u; \epsilon) = u + \epsilon\eta(x, u) + O(\epsilon^2) \end{aligned} \tag{3}$$

acting on an open subset $M \subset X \times U$, and the associated vector field, $\boldsymbol{v}$, defined as

$$\boldsymbol{v} = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}. \tag{4}$$

By appealing to the classical Lie point group of transformations, we may extend the vector field (4) from $X \times U$ to the jet-space $X \times U^{(n)}$

$$\boldsymbol{v}^{(n)} = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u} + \sum_{i=1}^{n} \eta^{(i)}\left(x, u, u_{(i)}\right)\frac{\partial}{\partial u_i}. \tag{5}$$

The contact condition (2) is preserved provided the $\eta^{(i)}$ satisfy the relations

$$\eta^{(i)}\left(x, u, u_{(i)}\right) = D_x\eta^{(i-1)} - u_i D_x\xi \qquad i = 1, \ldots, n \tag{6}$$

where D_X denotes the total derivative operator with respect to x.

The vector field (4) is associated with a point symmetry of (1) if and only if

$$\boldsymbol{v}^{(n)}(\Delta) = 0 \qquad \text{when} \qquad \Delta = 0. \tag{7}$$

On the other hand a function $I(x, u, u_1, \ldots, u_n) = c$, with c an arbitrary constant, is a first integral of the characteristic system

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}u}{\eta} = \frac{\mathrm{d}u_1}{\eta^{(1)}} = \cdots = \frac{\mathrm{d}u_1}{\eta^{(n)}} \tag{8}$$

if and only if

$$\boldsymbol{v}^{(n)}(I) \equiv 0. \tag{9}$$

Consequently, (7) ensures the existence of an analytic function σ such that $\sigma\Delta$ is a first integral of the characteristic system associated with (5). Indeed (7) may be rewritten as

$$\boldsymbol{v}^{(n)}(\Delta) = \mu\Delta \tag{10}$$

where the Lagrange multiplier μ has been introduced; on setting $\mu = \boldsymbol{v}^{(n)}(\sigma)/\sigma$, we obtain

$$\boldsymbol{v}^{(n)}(\sigma\Delta) \equiv \sigma\boldsymbol{v}^{(n)}(\Delta) + \Delta\boldsymbol{v}^{(n)}(\sigma) = 0. \tag{11}$$

Relation (7) is the computational procedure for finding the most general point symmetry group of an ordinary differential equation [5, 18]. Under the action of a symmetry group $\mathcal{G}$ any point on the surface $\mathcal{S}$ defined by the differential equation is moved along the manifold, i.e. any solution of the equation is transformed into another solution of the equation.

The vector field (5) is a *special* vector field that may be introduced in the jet-space $X \times U^{(n)}$. In fact, (5) is generated from the vector field (4) defined in $X \times U$, by a step by step prolongation in any subspace $X \times U^{(k)} \subset X \times U^{(n)}$, $(k = 1, 2, \ldots, n-1)$, which implies that ξ and η depend only on (x, u), whereas $\eta^{(k)}$ depends only on $(x, u, u_{(k)})$. The more general vector field that can be introduced in $X \times U^{(n)}$ is given by

$$\nu^{(n)} = \xi\left(x, u, u_{(n)}\right) \frac{\partial}{\partial x} + \eta\left(x, u, u_{(n)}\right) \frac{\partial}{\partial u} + \sum_{i=1}^{n} \zeta^{(i)}\left(x, u, u_{(n)}\right) \frac{\partial}{\partial u_i}. \tag{12}$$

A *generalized vector field,* whose definition is given in [5, 18], starts from the expression

$$\boldsymbol{v} = \xi\left(x, u, u_{(m)}\right) \frac{\partial}{\partial x} + \eta\left(x, u, u_{(m)}\right) \frac{\partial}{\partial u} \tag{13}$$

with $m \geqslant 1$. The prolongation of (13) in $X \times U^{(k)}$ is obtained again by means of formula (6). Because ξ, η and the various prolongations $\eta^{(k)}$ depend, respectively, on x, u and the derivatives up to the $k+m$ order, these structures are only *formally* vector fields and the corresponding $\boldsymbol{v}^{(k)}$ cannot be considered a *true* vector field in $X \times U^{(k)}$. The *true* structure of the vector field is preserved only for the special case $m = 1$ and for the constraint

$$\frac{\partial \eta}{\partial u_1} = u_1 \frac{\partial \xi}{\partial u_1} \tag{14}$$

(ensuring $\eta^{(1)} = \eta^{(1)}(x, u, u_1)$). The symmetries associated with a vector field (13) for $m = 1$ and satisfying (14) are termed *contact* or *dynamical* symmetries [5, 18].

The crucial role of symmetries in the study of ordinary differential equations is illustrated by a first-order equation possessing a point symmetry $\mathcal{G}$. Its solution may be obtained by quadrature using the canonical coordinates associated with $\mathcal{G}$. When $\Delta = 0$ is an nth-order equation $(n > 1)$ the differential invariants of $\mathcal{G}$ (i.e. the invariants of the extended group in $X \times U^{(n)}$) can be used to reduce the order of the equation by 1. For example, let $\Delta = 0$ be a second-order differential equation. The characteristic system associated with the prolongation of $\mathcal{G}$ in $X \times U^{(2)}$ is

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}u}{\eta} = \frac{\mathrm{d}u_1}{\eta^{(1)}} = \frac{\mathrm{d}u_2}{\eta^{(2)}}. \tag{15}$$

By virtue of relations (6), the complete set of functionally independent integrals for (15) is given by

$$w(x, u) = c_1 \qquad v(x, u, u_x) = c_2 \qquad \frac{\mathrm{d}v}{\mathrm{d}w} = c_3. \tag{16}$$

From (7) we conclude that $\sigma\Delta$ is an integral of (15) and therefore it must be given by

$$\sigma\Delta \equiv \Gamma(w, v, \mathrm{d}v/\mathrm{d}w) = 0 \tag{17}$$

i.e. the second-order equation $\Delta = 0$ is reduced to a first-order equation in the unknown v of the variable w.

The classical examples of reduction recorded in standard textbooks on differential equations are amenable to invariance arguments (see, for example, [14]) and help to understand the general belief that every ordinary differential equation which can be reduced to quadratures possesses the right number of symmetries. Recent apparent counterexamples due to González-López [8] emphasize the lack of a rigorous proof of this conjecture. Indeed the integrable equations reported in [8], even though not possessing classical Lie point symmetries, have a structure rich in *nonlocal symmetries* [2, 10].

The idea of nonlocal symmetries in the framework of ordinary differential equations may be traced to an example by Olver (exercise 2.30, p 184 of [18]), which indicates how to obtain an ordinary differential equation by symmetry reduction that may gain or lose symmetry with respect to the unreduced equation. This is usual when the symmetry algebra of the differential equation is nonsolvable. In [1] it is shown that, at least in some cases, these *hidden* (or *lost*) symmetries can be recovered as nonlocal symmetries, i.e. requiring that the infinitesimal generators in the vector field (4) may depend on an integral quantity (for example, $\int f(x,u)\,\mathrm{d}x$).

The aim of the present paper is to study the reduction of ordinary differential equations by a direct approach. We consider the standard procedure of reduction based on the differential invariants of a vector field. We characterize the necessary properties such that a vector field could be used to reduce a differential equation without considering invariance properties. Indeed the knowledge of symmetry groups for a given equation $\Delta = 0$ allows the reduction of the order *inter alia*. Symmetry groups form a powerful method for understanding the complete structure of a differential equation with potential application wider than just reduction of order by determination of a new system of coordinates.

In section 2 we determine the necessary properties of the first integrals of the characteristic systems (i.e. the differential invariants) *useful* for the reduction of ordinary differential equations. Then we give the characterization of all vector fields possessing the *right* characteristic system and study the link between these vector fields, nonlocal symmetries and λ-symmetries. Section 3 discusses some examples and section 4 contains concluding remarks.

2. Reduction of order

In the jet-space $X \times U^{(n)}$ we define as *telescopic* any vector field $\nu^{(n)}$ with associated characteristic system

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}u}{\eta} = \frac{\mathrm{d}u_1}{\zeta^{(1)}} = \cdots = \frac{\mathrm{d}u_n}{\zeta^{(n)}} \tag{18}$$

such that $\xi^2 + \eta^2 \neq 0$, and whose characteristic curves are given by the system of *independent* first integrals

$$\left.\begin{aligned} h_1(x,u,u_1) &= c_1 \\ h_2(x,u,u_1) &= c_2 \\ h_3(x,u,u_1,u_2) \equiv \frac{D_x h_2(x,u,u_1)}{D_x h_1(x,u,u_1)} &= c_3 \\ &\ \vdots \\ h_i(x,u,u_1,u_2,\ldots,u_{i-1}) \equiv \frac{D_x h_{i-1}(x,u,u_1,u_2,\ldots,u_{i-1})}{D_x h_1(x,u,u_1)} &= c_i \end{aligned}\right\} \tag{19}$$

where c_i are constants, $i = 1, \ldots, n+1$.

Telescopic vector fields are the key for the standard reduction method. For any differential equation $\Delta = 0$, which is an invariant of a telescopic vector field, the reduction procedure based on the differential invariants may be used. Indeed if $\Delta = 0$ is an invariant of $\nu^{(n)}$, possessing the complete set of independent integrals $h_1 = c_1, \ldots, h_{n+1} = c_{n+1}$, a first integral $\sigma\Delta$ must be [6]

$$\sigma\Delta \equiv \Gamma(c_1,\ldots,c_{n+1}) \equiv \Gamma\left(h_1, h_2, \frac{\mathrm{d}h_2}{\mathrm{d}h_1}, \frac{\mathrm{d}^2 h_2}{\mathrm{d}h_1^2}, \ldots, \frac{\mathrm{d}^{n-1} h_2}{\mathrm{d}h_1^{n-1}}\right) = 0. \tag{20}$$

Consequently, we have reduced the nth-order equation $\Delta = 0$ to an $(n-1)$th-order ordinary differential equation in the unknown h_2 of the variable h_1. On the other hand, when (20) is possible then Δ must be an invariant of a telescopic vector field.

Now we characterize the vector field $\nu^{(n)}$ such that the characteristic curves are given by (19). The vector fields can be defined only up to a multiplicative factor λ, since $\boldsymbol{v}$ and $\lambda\boldsymbol{v}$ have the same characteristic system.

Theorem 1. *In the jet-space $X \times U^{(n)}$ a vector field*

$$\nu^{(n)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \sum_{i=1}^{n}\zeta^{(i)}\frac{\partial}{\partial u_i}$$

is telescopic if and only if (up to a multiplicative factor)

$$\xi = \xi(x, u, u_1) \qquad \eta = \eta(x, u, u_1) \qquad \zeta^{(i)} = \zeta^{(i)}\left(x, u, u_{(i)}\right) \qquad i = 1, \ldots, n \tag{21}$$

where for $i = 2, \ldots, n$

$$\zeta^{(i)}(x, u, u_1, \ldots, u_i) = D_x\zeta^{(i-1)} - u_i D_x\xi + \frac{\zeta^{(1)} + u_1 D_x\xi - D_x\eta}{\eta - u_1\xi}\left(\zeta^{(i-1)} - \xi u_i\right). \tag{22}$$

Proof. Direct differentiation of equations (19) leads to (21)

$$h_1(x, u, u_1) = c_1 \qquad h_2(x, u, u_1) = c_2 \qquad h_n(x, u, u_1, \ldots, u_n) = c_n.$$

From $h_1 = c_1$ and $h_2 = c_2$, we have

$$\xi\frac{\partial h_1}{\partial x} + \eta\frac{\partial h_1}{\partial u} + \zeta^{(1)}\frac{\partial h_1}{\partial u_1} = 0 \tag{23}$$

and

$$\xi\frac{\partial h_2}{\partial x} + \eta\frac{\partial h_2}{\partial u} + \zeta^{(1)}\frac{\partial h_2}{\partial u_1} = 0. \tag{24}$$

The matrix

$$\begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial u_1} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial u_1} \end{pmatrix} \tag{25}$$

is of maximal rank because $h_1 = c_1, h_2 = c_2$ are independent. From (23) and (24), we conclude that ξ, η and $\zeta^{(1)}$ must depend, at most, on

$$\xi = \xi(x, u, u_1) \qquad \eta = \eta(x, u, u_1) \qquad \zeta^{(1)} = \zeta^{(1)}(x, u, u_1). \tag{26}$$

By differentiation of $h_i(x, u, u_1, \ldots, u_i) = c_i$, we have

$$\xi\frac{\partial h_i}{\partial x} + \eta\frac{\partial h_i}{\partial u} + \sum_{k=1}^{i}\zeta^{(k)}\frac{\partial h_i}{\partial u_k} = 0 \tag{27}$$

and, since $\frac{\partial h_i}{\partial u_i} \neq 0$, we may put

$$\zeta^{(i)} = -\frac{\xi\frac{\partial h_i}{\partial x} + \eta\frac{\partial h_i}{\partial u} + \sum_{k=1}^{i-1}\zeta^{(k)}\frac{\partial h_i}{\partial u_k}}{\frac{\partial h_i}{\partial u_i}}. \tag{28}$$

Then (21) follows.

To prove (22) and necessary conditions we consider the action of $\nu^{(n)}$ on $D_x h_i$, i.e.

$$\nu^{(n)}(D_x h_i) = \xi \frac{\partial D_x h_i}{\partial x} + \eta \frac{\partial D_x h_i}{\partial u} + \sum_{k=1}^{i} \zeta^{(k)} \frac{\partial D_x h_i}{\partial u_k}$$
$$= D_x \left(\xi \frac{\partial h_i}{\partial x} + \eta \frac{\partial h_i}{\partial u} + \sum_{k=1}^{i} \zeta^{(k)} \frac{\partial h_i}{\partial u_k} \right) - \frac{\partial h_i}{\partial x} D_x \xi - \frac{\partial h_i}{\partial u} \left(D_x \eta - \zeta^{(1)} \right)$$
$$- \sum_{k=2}^{i} \frac{\partial h_i}{\partial u_k} \left(D_x \zeta^{(k-1)} - \zeta^{(k)} \right) \tag{29}$$

and from (27) we have

$$\nu^{(n)}(D_x h_i) = -\frac{\partial h_i}{\partial x} D_x \xi - \frac{\partial h_i}{\partial u} \left(D_x \eta - \zeta^{(1)} \right) - \sum_{k=2}^{i} \frac{\partial h_i}{\partial u_k} \left(D_x \zeta^{(k-1)} - \zeta^{(k)} \right). \tag{30}$$

The functions $D_x h_i / D_x h_1$ $(i = 2, \ldots, n)$ are first integrals of the characteristic system (18) if and only if

$$\nu^{(n)}(D_x h_i) D_x h_1 - \nu^{(n)}(D_x h_1) D_x h_i = 0. \tag{31}$$

Use of (23), (30) and (31) leads to

$$\zeta^{(i)} = D_x \zeta^{(i-1)} - u_i D_x \xi + \frac{\zeta^{(1)} + u_1 D_x \xi - D_x \eta}{\eta - u_1 \xi} \left(\zeta^{(i-1)} - u_i \xi \right) \tag{32}$$

which completes the proof of the necessary condition. □

That (21) and (22) are sufficient conditions may be verified by a simple and direct computation based on the relationship

$$\nu^{(n)}(D_x h_i) = -\lambda \xi D_x h_i.$$

Relations (21) show that any telescopic $\nu^{(i)}$ is indeed a vector field in $X \times U^{(i)}$ $(i = 1, \ldots, n)$. Expression (22) determines $\zeta^{(i)}$ once the three arbitrary functions (26) are given. Therefore the most general telescopic vector fields are given up to within three arbitrary functions $\xi(x, u, u_1)$, $\eta(x, u, u_1)$, $\zeta^{(1)}(x, u, u_1)$ and up to a multiplicative factor.

Let us introduce the functions ρ_1 and ρ_2 defined by

$$\rho_1(x, u, u_1) = \frac{\zeta^{(1)} + u_1 \xi_x - \eta_x + u_1(u_1 \xi_u - \eta_u)}{\eta - u_1 \xi} \tag{33}$$

and

$$\rho_2(x, u, u_1) = \frac{u_1 \xi_{u_1} - \eta_{u_1}}{\eta - u_1 \xi}. \tag{34}$$

We can rewrite $\zeta^{(1)}$ as

$$\zeta^{(1)} = D_x \eta - u_1 D_x \xi + (\rho_1 + \rho_2 u_2)(\eta - \xi u_1) \tag{35}$$

and formulae (22) (for $i = 2, \ldots, n$) as

$$\zeta^{(i)} = D_x \zeta^{(i-1)} - u_i D_x \xi + (\rho_1 + \rho_2 u_2) \left(\zeta^{(i-1)} - \xi u_i \right). \tag{36}$$

Now the arbitrary elements are ξ, η and ρ_1.

The following trivial relationship between $\eta^{(i)}$ and $\zeta^{(i)}$ exists:

$$\begin{aligned} \zeta^{(1)} &= \eta^{(1)} + (\rho_1 + \rho_2 u_2)(\eta - \xi u_1) \\ \vdots \quad & \quad \vdots \\ \zeta^{(k)} &= \eta^{(k)} + (\rho_1 + \rho_2 u_2) \left(\zeta^{(k-1)} - \xi u_k \right). \end{aligned} \tag{37}$$

From (37) a direct relationship between telescopic vector fields and previously considered vector fields can be easily established. Indeed

- when $\rho_1 = \rho_2 = 0$ and $\xi_{u_1}^2 + \eta_{u_1}^2 \neq 0$ telescopic vector fields are the same as the vector fields of contact symmetries;
- when $\rho_1 = 0$ and $\xi_{u_1}^2 + \eta_{u_1}^2 = 0$ telescopic vector fields are the same as the vector fields of classical point symmetries;
- when $\rho_1 \neq 0$ and $\xi_{u_1}^2 + \eta_{u_1}^2 = 0$ telescopic vector fields are the same as the vector fields of λ-symmetries (introduced in section 2 of [16]).

We conclude that telescopic vector fields are more general than the vector fields already introduced in the literature in relation to classical invariance properties of differential equations.

When the arbitrary functions appearing in the definition of telescopic vector fields, $\nu^{(n)}$, are chosen such that $\sigma\Delta$ is a first integral of the corresponding characteristic system, we require that

$$\nu^{(n)}(\sigma\Delta) \equiv 0 \tag{38}$$

i.e.

$$\nu^{(n)}(\Delta) \equiv 0 \qquad \text{when} \qquad \Delta = 0. \tag{39}$$

Once an ordinary differential equation, $\Delta = 0$, is given and a solution, $\widehat{\xi}$, $\widehat{\eta}$, $\widehat{\rho}_1$ (or $\widehat{\zeta}_1$) of (38) is found it is possible to reduce $\Delta = 0$ by determining the solution of the characteristic subsystem consisting of the first two equations in (18). If $\widehat{\xi}, \widehat{\eta}$ do not depend on u_1, the algorithm is simplified because this subsystem consisting of the first two equations in (18) is uncoupled.

Telescopic vector fields seem to be the natural framework for the study of reduction methods based on differential invariants. In this respect it is very important to note that from the defining relation (32) of the $\zeta^{(i)}$ $(i = 2, \ldots, n)$ we may scale the three arbitrary functions by an arbitrary factor γ:

$$\left\{\xi, \eta, \zeta^{(1)}\right\} \to \left\{\gamma\xi, \gamma\eta, \gamma\zeta^{(1)}\right\} \tag{40}$$

and use definition (32) to obtain the following relation:

$$\zeta^{(i)}\left\{\gamma\xi, \gamma\eta, \gamma\zeta^{(1)}\right\} = \gamma\zeta^{(i)}\left\{\xi, \eta, \zeta^{(1)}\right\}. \tag{41}$$

Therefore the telescopic vector field ν associated with $\{\xi, \eta, \zeta^{(1)}\}$ and the vector field ν_γ associated with $\{\gamma\xi, \gamma\eta, \gamma\zeta^{(1)}\}$ must be related by

$$\nu_\gamma^{(n)} = \gamma\nu^{(n)}. \tag{42}$$

Because γ may depend on $u_{(r)}$, $(r \in \mathbb{N})$, (42) may be used to rewrite a given vector field in an equivalent format. This property may be used to endow a *formal* vector field (i.e. a generalized vector field that strictly speaking does not belong to the jet-space) with the format of a *true* vector field. For example, the nonlocal vector field named exponential by Olver in [18], is obtained on selecting

$$\gamma = \int P(x, u)\,\mathrm{d}x \qquad \xi = \xi(x, u) \qquad \eta = \eta(x, u) \tag{43}$$

and

$$\zeta^{(1)} = D_x\eta - u_1 D_x\xi + P(x, u)(\eta - \xi u_1) \tag{44}$$

and may be *pushed back* to the jet-space. If we consider the parametric form (35) this means $\rho_1 = P(x, u)$ and we recover the observation by Muriel and Romero on the relationship between an exponential nonlocal vector field and λ-symmetries [16]. In any case (43) and (44) allow a nice generalization of the exponential vector field that may be used to reduce an ordinary differential equation. Here, our more general dependences $\xi = \xi(x, u, u_1)$

and $\eta = \eta(x, u, u_1)$ give

$$P = \rho_1 + \rho_2 u_2 \tag{45}$$

where $\rho_1(x, u, u_1)$ is arbitrary and ρ_2 is as in (34).

3. Examples

The first example is about the differential equation

$$\Delta \equiv 2xu_1u_2 - (x+1)u_1^2 + 2x^2(u - u_1) = 0. \tag{46}$$

By computation of

$$\nu^{(2)}(\Delta) \equiv 0$$

we obtain

$$\left[2u_1u_2 - u_1^2 + 4x\,(u - u_1)\right]\xi + 2x^2\eta + 2\left(xu_2 - (x+1)u_1 - x^2\right)\zeta^{(1)} + 2xu_1\zeta^{(2)} = 0. \tag{47}$$

In (47) $\zeta^{(2)}$ is defined in (36) for $i = 2$.

It is possible to check that the telescopic vector field with components

$$\xi \equiv 0 \qquad \eta = u_x \qquad \zeta^{(1)} = x \tag{48}$$

and

$$\zeta^{(2)} = 1 + \frac{x - u_2}{u_1}x \tag{49}$$

is a simple solution of (47).

The vector field (48) can neither be associated with a point symmetry nor with a λ-symmetry, because $\partial\eta/\partial u_x \neq 0$ and also cannot be associated with a generalized symmetry because $\zeta^{(1)} \neq \eta^{(1)}$. In any case, computing the differential invariants of (48) as

$$x = h_1 \qquad xu - \frac{u_1^2}{2} = h_2$$

equation (46) is reduced to

$$h_1\frac{\mathrm{d}h_2}{\mathrm{d}h_1} - (h_1 + 1)h_2 = 0.$$

The second example is more involved. In two very recent papers [8, 17] the integration of the equation

$$u_3 + uu_2 - \tfrac{3}{2}u_1^2 = 0 \tag{50}$$

is considered. Equation (50) is a special case of the Chazy equation

$$u_3 + uu_2 + ku_1^2 = 0 \tag{51}$$

and admits three Lie point symmetries

$$\boldsymbol{v}_1 = \frac{\partial}{\partial x} \qquad \boldsymbol{v}_2 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} \qquad \boldsymbol{v}_3 = x^2\frac{\partial}{\partial x} + (12 - 2xu)\frac{\partial}{\partial u} \tag{52}$$

which form a representation of the *nonsolvable* algebra $sl(2, R)$.

When a third-order equation is invariant under the action of a three-dimensional *solvable* Lie algebra it is possible by standard methods to reduce it to first order, because the symmetries are conserved in the reduction procedure [5, 18]. In the case of (50) because the algebra generated by (52) is nonsolvable, standard methods of reduction fail, since, after the first reduction, at least one symmetry obtained by the *remaining* generators disappears. In [17] it is shown how these generators may be recovered by means of a *new class of symmetries*, introduced in [16] and named λ-symmetries. The Chazy equation then may be reduced step by step. On the other hand, the *remaining* generators may be recovered by considering [7] nonlocal symmetries (of exponential type) enabling the Chazy equation again to be reduced step by step. Moreover, the initiation of the process of order reduction need not be a point (or a contact) symmetry. Instead, it is possible to start the reduction with a nonlocal symmetry or λ-symmetry.

For example, when $k = 1$, (51) admits the nonlocal symmetries related to the vector fields (see [8])

$$\begin{aligned} \boldsymbol{v}_4 &= \exp\left[-\int u\,\mathrm{d}x\right]\frac{\partial}{\partial u} \\ \boldsymbol{v}_5 &= \exp\left[-\int u\,\mathrm{d}x\right]\int\exp\left[\int u\,\mathrm{d}x\right]\frac{\partial}{\partial u} \\ \boldsymbol{v}_6 &= \exp\left[-\int u\,\mathrm{d}x\right]\exp\left[x\int u\,\mathrm{d}x\right]\frac{\partial}{\partial u}. \end{aligned} \tag{53}$$

In the special case considered here the vector fields (53) may be easily determined. Let us consider

$$\boldsymbol{v} = \eta\frac{\partial}{\partial u} \tag{54}$$

whose prolongation is

$$\boldsymbol{v}^{(3)} = \eta\frac{\partial}{\partial u} + D_x(\eta)\frac{\partial}{\partial u_1} + D_{xx}(\eta)\frac{\partial}{\partial u_2} + D_{xxx}(\eta)\frac{\partial}{\partial u_3}. \tag{55}$$

On expanding

$$\boldsymbol{v}^{(3)}\left(u_3 + uu_2 + ku_1^2\right) = 0$$

we obtain

$$D_{xxx}(\eta) + D_{xx}(\eta)u + \eta u_2 + 2D_x(\eta)u_1 = 0 \tag{56}$$

or in more compact form

$$D_{xx}[D_x(\eta) + \eta u] = 0. \tag{57}$$

The determining equation (56) has been obtained for an arbitrary dependence of η. It is easy to check that the vector fields (53) are trivial solutions of (57). We point out that equation (57) is not restricted with respect to the manifold of the solutions of equation (51).

By using $\boldsymbol{v}_4$, which is a telescopic vector field, and the differential invariants

$$y = x \qquad v = u_1 + \tfrac{1}{2}u^2$$

it is possible to reduce (51) to the simple equation $\mathrm{d}^2v/\mathrm{d}u^2 = 0$.

Since $\boldsymbol{v}_4$ is a telescopic vector field, the approaches in [8, 17] must be equivalent. To show this we consider the equation for telescopic symmetries of (51), i.e.

$$\zeta^{(3)} + \eta u_2 + u\zeta^{(2)} + 2u_1\zeta^{(1)} = 0. \tag{58}$$

It is easy to check that

$$\xi(x,u) \equiv 0 \qquad \lambda = -u \qquad \eta = 1 \tag{59}$$

and that the corresponding prolongations

$$\zeta^{(1)} = -u \qquad \zeta^{(2)} = -u_1 + u^2 \qquad \zeta^{(3)} = -u_2 + 3uu_1 - u^3 \tag{60}$$

are a solution of (58). It is clear on choosing $\gamma = \exp\left[-\int u\,\mathrm{d}x\right]$ that (59) is the λ-symmetry vector field corresponding to $\boldsymbol{v}_4$.

The vector fields $\boldsymbol{v}_5$ and $\boldsymbol{v}_6$ are not telescopic vector fields and appear to have no role in reducing the order of (51).

4. Concluding remarks

Theorem 1 extends the vector field in the jet-space for the purpose of reducing the order of an ordinary differential equation. Our paper is clearly related to that by Muriel and Romero on λ-symmetries [16], but as already stated telescopic vector fields are more general and our direct approach establishes not only a sufficient condition (as in [16]) but also a *necessary* condition.

Moreover, we have clarified the relationship of telescopic vector fields and nonlocal symmetries and emphasized that symmetries effective in the reduction of ordinary differential equations can be rewritten as telescopic vector fields. This does not mean that telescopic vector fields are strictly equivalent to nonlocal symmetries (which were introduced many years before the telescopic vector fields and λ-symmetries). The main difference is about their determination. It is well known that there is no systematic approach to the determination of nonlocal symmetries, whereas we have an efficient algorithm to compute telescopic vector fields. This, obviously, does not mean that we are always able to solve the determining equations of telescopic vector fields for which an ordinary differential equation is a relative invariant, but this situation occurs also for classical Lie symmetries of differential equations (for example, the simple case of first-order differential equations).

Telescopic vector fields enlarge the study of the integrability of differential equations. The interesting observations on nonlocal and contact symmetries done in several papers [1, 3, 7, 9, 10, 19] can be related to telescopic vector fields. For example, it is well known that any ordinary differential equation of the first order is invariant under the action of a Lie point symmetry group (i.e. any first-order ordinary differential equation possesses an integrating factor) [5, 18]. As regards a second-order differential equation, let us consider

$$u_2 = E(x, u, u_1)$$

whose determining equation for the admitted telescopic vector fields is given by

$$\left(\zeta^{(2)} - E_{u_1}\zeta^{(1)} - E_u\eta - E_x\xi\right)\Big|_{u_2=E} \equiv 0. \tag{61}$$

This relationship contains at most the first derivative of the unknown u. Because in (61) we have to determine the arbitrary element $\zeta^{(1)}(x, u, u_1)$ any second-order equation may be (theoretically) reduced by a telescopic vector field with $\xi(x, u)$, $\eta(x, u)$. This agrees with the results in [9] (section 5, p 5357) and with the results of [15, 19] where it is shown that all second-order differential equations are invariant under contact symmetries. Indeed all contact symmetries are associated with telescopic vector fields, but working with telescopic vector fields enables the components ξ and η to be chosen such that the computation of the differential invariants is simplified.

A final remark is that a more geometric theory of telescopic vector fields and λ-symmetries is surely possible by means of the theory of solvable structures [4, 11] or the theory of coverings [12]. The direct approach has been adopted here because of its simpler value and accessibility to a wider readership.

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